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Chiral primaries in the Leigh-Strassler deformed $\mathcal{N} = 4$ SYM — a perturbative study

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Abstract: We look for chiral primaries in the general Leigh-Strassler deformed $\mathcal{N} = 4$ super Yang-Mills theory by systematically computing the planar one-loop anomalous dimension for single trace operators up to dimension six. The operators are organised into representations of the trihedral group, $\Delta(27)$, which is a symmetry of the Lagrangian. We find an interesting relationship between the $\mathrm{U}(1)_R$-charge of chiral primaries and the representation of $\Delta(27)$ to which the operator belongs. Up to scaling dimension $\Delta_0 = 6$ (and conjecturally to all dimensions) the following holds: The planar one-loop anomalous dimension vanishes only for operators that are in the singlet or three dimensional representations of $\Delta(27)$. For other operators, the vanishing of the one-loop anomalous dimension occurs only in a sub-locus in the space of couplings.

Keywords: AdS-CFT Correspondence, Supersymmetry and Duality, Supersymmetric gauge theory.

Contents

1. Introduction

The AdS-CFT correspondence due to Maldacena [1] is a concrete realisation of 't Hooft's proposal relating Yang-Mills (at large-N) to string theory. The correspondence relating $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory to type IIB strings propagating on $AdS_5 \times S^5$ has been tested in many different ways. The first tests involved matching computations in the supergravity limit of the string theory and the corresponding ones in the conformal field theory (CFT) following the proposal in [2]. Recently, it was realised that the anomalous dimensions of operators in this theory are given by the energy spectrum of a spin-chain [3, 4].

The $\mathcal{N} = 4$ theory has a high degree of symmetry and thus it is of interest to understand versions with lesser symmetry. Orbifolds of this theory provide one obvious class of CFT's [5, 6]. These lead to theories which are dual to type IIB string theory propagating on $AdS_5 \times X^5$, where X^5 are five manifolds that are orbifolds of S^5. More generally, X^5 must be a Sasaki-Einstein manifold. A new class of such manifolds that lead to CFT's with $\mathcal{N} = 1$ supersymmetry have also been constructed recently. These manifolds have been called L_{pqr} spaces — cones over these spaces are Ricci-flat non-compact six dimensional manifolds with SU(3) holonomy and are natural generalisations of the conifold [7, 8]. Thus, these are examples where both sides of the AdS-CFT correspondence are understood even though we do not fully understand string theory on AdS spaces.

From a field theoretic perspective, Leigh and Strassler [9] considered (multi-parameter) marginal deformations of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory that preserve $\mathcal{N} = 1$ supersymmetry and are conformal. We shall refer to these theories are known as the Leigh-Strassler (LS) theories. While there have been attempts [10] to understand the anticipated dual string theories for these field theories, the precise correspondence is not known in all generality. An important development due to Lunin and Maldacena [11] was the construction of the gravity duals for the β-deformed $\mathcal{N} = 4$ SYM which is a sub-class in the generalised LS family of $\mathcal{N} = 1$ theories. Rational values of $\beta = m/n$ in the deformation turn out to be related to $(\mathbb{Z}_n \times \mathbb{Z}_n)$ orbifolds of S^5 with discrete torsion [12 − 15].

The most general Leigh-Strassler $\mathcal{N} = 1$ preserving deformations of $\mathcal{N} = 4$ SYM is given (in superfields) by the following superpotential:

$$W = ih \ \mathrm{Tr}\Big(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\Big) + \frac{ih'}{3} \ \mathrm{Tr}\Big(\Phi_1^3 + \Phi_2^3 + \Phi_3^3\Big) \ . \tag{1.1}$$

The $\mathcal{N} = 4$ limit occurs when $h = g$ and $\beta = h' = 0$. The β-deformed theory is obtained when $\beta \neq 0$ and $h' = 0$. For generic values of the deformations, the global symmetry of the deformed theory is the trihedral group $\Delta(27)$ with its centre being a $\mathbb{Z}_3$ sub-group of the $\mathrm{U}(1)_R$ symmetry. This is in contrast to the situation when $X^5 = L_{pqr}$, where one has a $\mathrm{U}(1)^3$ symmetry which is intimately related to the toric nature of these spaces.

The LS deformations are parametrised by the four couplings: g (the Yang-Mills coupling constant), h, h' and $q \equiv e^{i\pi\beta}$. While these are all marginal at the classical level, they are not all marginal in the quantum theory. However, it has been argued by Leigh and Strassler that in a subspace of the four-dimensional space of couplings, the theory is conformal. In particular, the vanishing the four beta functions is related to the vanishing of the anomalous dimension of the scalars fields. The exact expression for the subspace is not known. However, it is known to two loops and is given by

$$|h|^2\Big(1 \ + \ \frac{1}{N^2}(q-\bar{q})^2\Big) \ + \ |h'|^2\frac{N^2-4}{2N^2} \ = \ g^2 \ . \tag{1.2}$$

In the large N limit that we pursue in this paper, the above condition simplifies to

$$|h|^2 + \frac{|h'|^2}{2} = g^2 \ . \tag{1.3}$$

We also choose to work with real β which is possible at the one-loop level. In principle, this can be done at higher loops as well since the imaginary part of β can be gotten rid of by redefining h.

Perturbative studies of the β-deformed theory have been carried out by several authors [16 − 21]. Chiral primaries of these theories are known well at least in the planar (large-N) case [17]. The ultraviolet finiteness of the β-deformed theory has been studied in ref. [22] and a proof of its ultraviolet finiteness at the planar level has also appeared recently [23, 24].

For the general LS deformations, the anomalous dimensions for chiral operators with low values of scaling dimension has been carried out (using super-graphs) in ref. [21] as well as ref. [25]. This paper focuses on a planar one-loop computation of anomalous dimensions of single trace operators in order to systematically search for operators that are protected in the LS theory.[1] In this paper, we systematically search for operators whose anomalous dimensions vanish at planar one-loop in the LS theory. The operators are organised into representations of $\Delta(27)$. We find that protected operators appear in one of the three representations, $\mathcal{L}_{0,0}$ or $\mathcal{V}_a$ depending on the value of the scaling dimension, Δ_0 modulo three.

The paper is organised as follows. In section two, we present some of the background needed for the paper. In particular, we present the F-term superpotential in component form and show that it contains double trace operators. In section three, we compute the anomalous dimensions for operators of the form $\mathrm{Tr}(Z_1^k Z_2^l Z_3^m)$. We present the details of our computation as well as verify that all contributions that appear from non F-term interactions cancel and that our results are gauge independent. In section four, we compute the anomalous dimensions for operators up to dimension six in the β-deformed theory as a preliminary to the computing in the general Leigh-Strassler theory. We verify that the results are consistent with expected results up to dimension six operators. In section five which contains the main results of this paper, after organising the operators using the trihedral group, we compute the planar one-loop anomalous dimension in the Leigh-Strassler theory. We present our conclusions and outlook in section six. Some technical details are relegated to the appendix for completeness.

2. Background

2.1 The component Lagrangian for the LS theory

We present here the details of the Lagrangian for the general Leigh-Strassler deformation in component form. It turns out that unlike the $\mathcal{N} = 4$ Lagrangian, this Lagrangian *cannot* be written in terms of a single trace though the superfield Lagrangian is written as a single

[1] The planar condition simplifies things and enables us to obtain concrete results for operators with dimensions up to six. The complexity arises from the increase in the number of operators that one has to consider. For instance, at $\Delta_0 = 6$ and $Q = 0$, one has to consider 46 operators. It must be pointed out that for the β-deformed theory in the planar limit, ref. [26] has obtained an integrable dilatation operator and diagonalized it using a Bethe ansatz.

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trace.[2] There are some terms that can only be written as a double trace — these terms are however suppressed by a power of $1/N$ but cannot be neglected in the large N limit as we will see.

The simplest way to see the appearance of double trace operators is to consider trace identity (for SU(N) generators in the fundamental representation):

$$\mathrm{Tr}(AT^a)\mathrm{Tr}(BT^a) = \mathrm{Tr}(AB) - \frac{1}{N}\mathrm{Tr}(A)\mathrm{Tr}(B) \tag{2.1}$$

Notice that both $F_i^a = \partial\bar{W}/\partial\bar{Z}_i^a$ and $\bar{F}_i^a = \partial W/\partial Z_i^a$ are both of the form $\mathrm{Tr}(AT^a)$ for some A. We thus see that $|F_i^a|^2$ cannot be written in single trace form (using the above identity) unless the operators A and B are traceless (as in the $\mathcal{N}=4$ limit). The F-term interactions (involving bosonic fields) for the LS theory is given by the potential[3]

$$\begin{aligned} V_F(Z) = \mathrm{Tr}\Big(&|h'|^2\bar{Z}_1{}^2 Z_1^2 + \ h\bar{h}'[Z_2,Z_3]_q\bar{Z}_1{}^2 - \bar{h}h'[\bar{Z}_2,\bar{Z}_3]_q Z_1^2 - |h|^2[Z_2,Z_3]_q[\bar{Z}_2,\bar{Z}_3]_q\Big) \\ -\frac{1}{N}\Big[&|h'|^2\,\mathrm{Tr}(\bar{Z}_1{}^2)\,\mathrm{Tr}(Z_1^2) + \ h\bar{h}'\,\mathrm{Tr}([Z_2,Z_3]_q)\,\mathrm{Tr}(\bar{Z}_1{}^2) - \bar{h}h'\,\mathrm{Tr}(Z_1^2)\,\mathrm{Tr}([\bar{Z}_2,\bar{Z}_3]_q) \\ -&|h|^2\,\mathrm{Tr}([Z_2,Z_3]_q)\,\big(\mathrm{Tr}[\bar{Z}_2,\bar{Z}_3]_q\big)\Big] + \text{cyclic permutations} \end{aligned} \tag{2.2}$$

The first line are the single trace operators while the last two lines are the double trace operators. When $h'=0$, one can see that the double trace terms are proportional to $(q-\bar{q})^2$ which vanishes when $q=\pm1$. Thus the $\mathcal{N}=4$ SYM theory does not have double trace terms in its component Lagrangian. Note that the double trace operators also do not exist when the gauge group is U(N). Since the D-terms are unaffected by the Leigh-Strassler deformations to the superpotential, they are the identical to the one in the $\mathcal{N}=4$ theory (written in terms of $\mathcal{N}=1$ superfields). The detailed Lagrangian is given in appendix A.

2.2 Symmetries of the LS theory

The $\mathcal{N}=4$ SYM theory has a R-symmetry which is SU(4). In the β-deformed theory, this is broken down to $\mathrm{U}(1)^3$ — each of the three scalars has charge one under only one of three U(1)'s with $\mathrm{U}(1)_R$ being identified with the diagonal. In the Leigh-Strassler theory, the $\mathrm{U}(1)^3$ is further broken down to $\mathrm{U}(1)_R\times\mathbb{Z}_3$. The LS theory has another symmetry given by the cyclic permutation $\mathcal{C}_3$ of the three scalar fields. This however, does not commute with the $\mathrm{U}(1)_R\times\mathbb{Z}_3$. The trihedral group, $\Delta(27)\sim(\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathcal{C}_3$, is a discrete subgroup of $\mathrm{SU}(3)\subset\mathrm{SU}(4)$ that captures the essential non-abelian nature. The centre of this group is a $\mathbb{Z}_3\subset\mathrm{U}(1)_R$. The $\mathrm{U}(1)_R$ charge which is proportional to the scaling dimension for chiral primaries becomes a $\mathbb{Z}_3$ valued charge. Specifically, in our conventions, the value of the scaling dimension, Δ_0, modulo three is the $\mathbb{Z}_3$ charge. Historically, the appearance of a 27 parameter non-abelian discrete subgroup was first noticed in ref. [27].[4] Our attention to the appearance of the $\Delta(27)$ was drawn from ref. [28] which attributed it to S. Benvenuti.

[2]This result is implicitly present in the work of Freedman and Gürsoy [17].

[3]In this paper, we denote the bosonic component of the superfield Φ_i by Z_i and the q-deformed commutator is $[Z_1,Z_2]_q\equiv qZ_1Z_2-\bar{q}Z_2Z_1$.

[4]We thank Ofer Aharony for bringing this to our notice.

We have found the trihedral group extremely useful in organising the chiral operators. For instance, it reduced the number of free parameters for an operator at dimension six from 46 to three different operators with 26, 10 and 10 parameters.

One may ask whether $\Delta(27)$ remains a symmetry of the quantum theory.[5] As argued by Leigh and Strassler, the full quantum effective potential will continue to be of the same form as the classical one. In other words, quantum corrections renormalised the coefficients h, h' and β. Thus, $\Delta(27)$ will remain a symmetry of the superpotential. For $\Delta(27)$ to be a symmetry in the quantum theory, however it is also necessary that the Kähler potential also respects this symmetry. The tree-level Kähler potential does respect the symmetry and one needs to check if this remains true at higher orders in perturbation theory. We have not resolved this issue completely and hope to report on this in the future. Since our one-loop computation makes use of only tree-level interactions, the use of $\Delta(27)$ is a valid one.

Another useful invariance of the action is the following:

$$\Phi_1 \leftrightarrow \Phi_2\,,\ h \to -h\,,\ \beta \to -\beta \text{ and } h' \to h'\ . \tag{2.3}$$

This is *not* a symmetry since it acts on the couplings as well. This however leads to restrictions on the possible renormalisation of coupling constants.

2.3 Propagators

The propagators for the various fields are as follows, where Feynman gauge has been chosen to write down the gauge propagator.

$$\begin{aligned} \langle Z_i^a \bar{Z}_j^b \rangle &= \delta_{ij}\delta^{ab}\frac{1}{k^2}\,, & \langle A_\mu^a A_\nu^b \rangle &= -\delta^{ab}\frac{g_{\mu\nu}}{2k^2}\,, \\ \langle \lambda^a \bar{\lambda}^b \rangle &= -\delta^{ab}\,\frac{\sigma^\mu k_\mu}{2k^2}\,, & \langle \psi_i^a \bar{\psi}_j^b \rangle &= -\delta^{ab}\delta_{ij}\,\frac{\sigma^\mu k_\mu}{k^2}\ . \end{aligned} \tag{2.4}$$

We have explicitly verified the gauge independence of our results by working in the Landau gauge as well.

3. Anomalous dimension of $\mathrm{Tr}\big(Z_1^k Z_2^l Z_3^m\big)$

In $\mathcal{N} = 1$ gauge theories, it is known that holomorphy is the basis for certain non-renormalisation theorems [29]. In order to prove properties that make use of holomorphy, one usually works in superfields and regularisation schemes that are compatible with holomorphy.[6] In our context where we are computing anomalous dimensions of operators involving scalars that arise from chiral superfields, holomorphy implies that the only interaction terms that contribute to the anomalous dimension are those that arise from F-terms as we will explicitly verify.

[5] We thank Justin David for raising this question.

[6] A much more modern use of holomorphy and its relation to the Wilsonian effective action is due to Seiberg [30].

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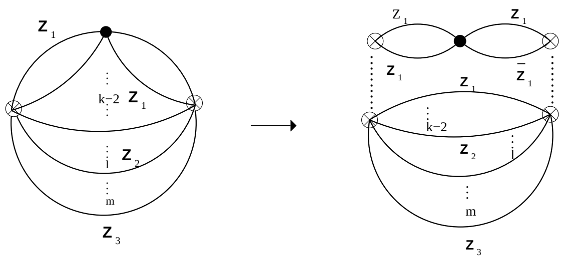

Figure 1: Contribution from $\mathrm{Tr}\big([Z_1,\bar{Z}_1][Z_1,\bar{Z}_1]\big)$. The figure to the right schematically shows how the logarithmic divergence was extracted. The interaction vertex is labelled by a filled-in circle.

In computing the anomalous dimension of the operator $\mathcal{O}$, we compute the two-point function of this operator with its conjugate operator, which we denote by $\bar{\mathcal{O}}$ and study its singularity when the two operators are coincident. One expects

$$\lim_{|x|\to 0}\langle \mathcal{O}(x)\bar{\mathcal{O}}(0)\rangle \sim \frac{1}{|x|^{2\Delta_0}} - \frac{\gamma \log|x|^2}{|x|^{2\Delta_0}}\,,$$

where Δ_0 is the naive scaling dimension of operator and γ its anomalous dimension. Thus the anomalous dimension is computed extracting the logarithmic singularities and summing over all such contributions.

For the family of operators $\mathrm{Tr}\big(Z_1^k Z_2^l Z_3^m\big)$, we find that, at large N (i.e., in the planar limit), the one-loop contribution to the anomalous dimension from all interactions take the following form (on using dimensional regularisation):

$$\frac{N^{k+l+m+1}}{256\pi^6|x|^{2(k+l+m)}}\left(\frac{1}{\epsilon}+3\log|x|^2+\text{constant}\right)\times \text{a combinatoric factor} \tag{3.1}$$

When the sum of all contributions is such that the coefficient of $\ln|x|^2$ vanishes, we obtain a candidate for the chiral primary. Recall that for chiral primaries, the scaling dimension is determined entirely by its $\mathrm{U}(1)_R$-charge and hence should receive *no* corrections. For a true chiral primary, γ vanishes to all orders. So the vanishing of the planar one-loop contribution to any operator does not imply that it is a chiral primary since it could obtain contributions at higher orders. However, such operators provide us with candidates for chiral primaries.

3.1 Cancellation of non F-term contributions

Since we are working in component form, we need to explicitly verify that all non-holomorphic contributions to the anomalous dimensions of chiral fields cancel out. These contributions should vanish irrespective of whether the operator is a chiral primary or not. While this is expected [31], we use this computation as a non-trivial check of our results. Such contributions come from three kinds of terms:

- **D-term:** figures 1 and 2 arise from the D-term interaction vertex $-(g^2/4)\sum_{i,j}\mathrm{Tr}\big([Z_i,\bar{Z}_i][Z_j,\bar{Z}_j]\big)$.

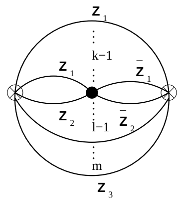

Figure 2: Contribution from $\mathrm{Tr}\big([Z_1,\bar{Z}_1][Z_2,\bar{Z}_2]\big)$

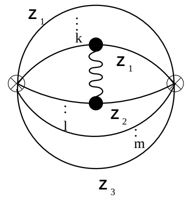

Figure 3: Contribution from gluon exchange

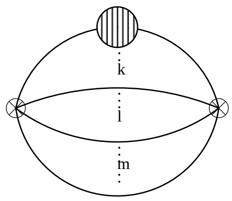

Figure 4: Contribution from corrections to the scalar propagator

- **Gluon exchange:** figure 3 indicates the contribution from the gluon-scalar interaction vertex $ig\mathrm{Tr}\big(\partial_\mu Z_i[A^\mu,\bar{Z}_i]+\partial_\mu\bar{Z}_i[A^\mu,Z_i]\big)$. This diagram is gauge dependent and is logarithmically divergent in the Feynman gauge, but non-divergent in the Landau gauge.

- **Self-energy:** figure 4 indicates the contribution arising from the self-energy correction to all scalar propagators. This one is also a gauge dependent contribution.

As we will see the three contribution cancel for all operators that we are considering here. We now provide some details of this cancellation. We have also verified that the cancellation holds in both the Feynman and Landau gauge though we will provide details for the Feynman gauge.

3.2 Details of the cancellation

Evaluation of figure 1. Figure 1 has contributions coming from the interaction term $-\frac{g^2}{4}\mathrm{Tr}\big([Z_1,\bar{Z}_1][Z_1,\bar{Z}_1]\big)$. We evaluate the loop correction by doing one momentum integral and then taking the inverse Fourier transform to get the answer in position space. We consider the fields pairwise and find the loop correction due to the interaction. We calculate loops involving interaction vertex separately in momentum space and then Fourier transform to position space. We multiply this with the contribution $1/|x|^{2(k+l+m-2)}$ from the part which does not involve interaction vertex. This is schematically explained in the diagram on the right in figure 1

For an operator of general form $\mathrm{Tr}\big(Z_1^k Z_2^l Z_3^m\big)$, there are $(k-1)$ contributions from this vertex. But when we have fields of only one flavor, say Z_1, there is an additional term giving a total of k from this interaction vertex. However when $k=1$, there are no contributions from this vertex. Similar contributions for fields of other two flavors ensures that the combinatoric factor is symmetric in k, l, m. Hence the contribution from this diagram is

$$\frac{2g^2 \cdot N^{k+l+m+1}\big(J_{klm}+G_{klm}\big)}{4|x|^{2(k+l+m-2)}}\left[\int \frac{d^Dp}{(2\pi)^D} e^{ip\cdot x}\left(\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k-p)^2}\right)^2\right] \tag{3.2}$$

where G_{klm} and J_{klm} are the combinatorial factors with all the properties described above, given by (δ_m is the Kronecker delta function and is non-vanishing only when $m=0$)

$$G_{klm} = -3 + \big(\delta_k + \delta_l + \delta_m\big) + \big(\delta_k\delta_l + \delta_l\delta_m + \delta_m\delta_k\big) - 3\delta_k\delta_l\delta_m \tag{3.3}$$

and

$$J_{klm} = k + l + m - \big(\delta_k\delta_l\delta_{m-1} + \delta_{k-1}\delta_l\delta_m + \delta_{l-1}\delta_m\delta_k\big) \tag{3.4}$$

The momentum integral in eq. (3.2) is evaluated in appendix. We find the contribution from this figure as

$$\begin{aligned} &\frac{g^2 N^{k+l+m+1}\big(J_{klm}+G_{klm}\big)}{2(|x|^2)^{k+l+m-2}} \frac{\Gamma^2[\epsilon]\Gamma^4[1-\epsilon]}{(4\pi)^D\Gamma^2[2-2\epsilon]}\left[\int \frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot x}}{(p^2)^{2\epsilon}}\right] \\ &\qquad = \frac{g^2 N^{k+l+m+1}\big(J_{klm}+G_{klm}\big)}{256\pi^6|x|^{2(k+l+m)}}\left(\frac{1}{\epsilon} + 1 + 5\gamma_E + 3\log\pi + 3\log|x|^2\right), \end{aligned} \tag{3.5}$$

where γ_E is Euler constant.

Evaluation of figure 2. Figure 2 involves the D-term interaction$-\frac{g^2}{4}\mathrm{Tr}\big([Z_1,\bar{Z}_1][Z_2,\bar{Z}_2]\big)$ and similar terms obtained by cyclic permutation of flavor indices. Here we notice that when the operator has fields of all three flavors the combinatorial factor must be 3. When the there are fields of two flavors this factor must be 2. There is no such interaction when there are only fields of single flavor and hence there is no contribution from this diagram. The integral to be evaluated is the same as in figure 1. The contribution from this interaction vertex is

$$-g^2\,\frac{N^{k+l+m+1}\ G_{klm}}{256\pi^6(|x|^2)^{k+l+m}}\left(\frac{1}{\epsilon} + 1 + 5\gamma_E + 3\log\pi + 3\log|x|^2\right) \tag{3.6}$$

Contribution from figure 3. The interaction vertex is $ig\ \mathrm{Tr}\partial_\mu Z[A^\mu,\bar{Z}] + \text{c.c.}$ The calculation of corrections is again done by taking propagators pairwise, as in the previous cases. We realise that out of the different types of contractions possible only two are giving rise to any divergence. To find the combinatorial factor for figure 3 can be easily identified

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as the some of the combinatorial factors of the above two diagrams. The net contribution from this diagram is

$$2\frac{g^2}{2}\,\frac{N^{k+l+m+1}J_{klm}}{(|x|^2)^{k+l+m-1}}\times\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k\cdot(k-p)}{(k-q)^2(k-p)^2q^2(q-p)^2k^2}$$
$$=\frac{g^2N^{k+l+m+1}J_{klm}}{256\pi^6(|x|^2)^{k+l+m}}\times\left(\frac{1}{\epsilon}+2+3\gamma_E+3\log\pi+3\log|x|^2\right) \tag{3.7}$$

Contribution from the self-energy. This contribution arises out of the one-loop correction to the scalar propagator $\langle Z_i\bar{Z}_i\rangle$. The calculation of one-loop corrected scalar propagator is given in the appendix. This correction is represented by the blob in figure 4. Here again we calculate the contribution from figure 4 by taking lines pairwise, where one of the two lines has the blob. (This is needed to match the numerical factors in the calculation of other diagrams.) This blob can appear on any of the $\big(k+l+m\big)$ lines. Taking into consideration that the gauge group is SU(N) the combinatorial factor from figure 4 is seen to be J_{klm}. Multiplying this by a factor of $\frac{1}{|x|^{2(k+l+m-2)}}$ from the rest of the $(k+l+m-2)$ lines, the one-loop correction to scalar propagator is obtained in momentum space from eq. (D.7) as

$$-2N\big(|h|^2+|h'|^2/2\big)\mathrm{Tr}(T^aT^b)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k-p)^2p^2}$$
$$=-2N\big(|h|^2+|h'|^2/2\big)\mathrm{Tr}(T^aT^b)\frac{\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}(p^2)^{1+\epsilon}} \tag{3.8}$$

Inserting this into the figure 4 and calculating the loop

$$-2\frac{N^3\big(|h|^2+|h'|^2/2\big)\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(p-q)^2(p^2)^{1+\epsilon}}$$
$$=-2\frac{N^3\big(|h|^2+|h'|^2/2\big)\Gamma(\epsilon)\Gamma(2\epsilon)B[1-2\epsilon,1-\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{4-2\epsilon}\;B[1,1+\epsilon]\;\Gamma(2+\epsilon)\;(q^2)^{2\epsilon}} \tag{3.9}$$

The N^3 factor arises when we contract the $\mathrm{Tr}(T^aT^b)$ with generators coming from the operator. In the above, the momentum integral is again evaluated using Feynman parametrisation and then Fourier transformed to position space to obtain

$$-2\frac{N^3\big(|h|^2+|h'|^2/2\big)\;J_{klm}}{256\pi^6|x|^4}\left(\frac{1}{\epsilon}+2+3\gamma_E+3\log\pi+3\log|x|^2\right) \tag{3.10}$$

Together with the rest of the lines in figure 4, which gives a factor of $\frac{N^{k+l+m-2}}{|x|^{2(k+l+m-2)}}$ multiplying it, the total contribution of figure 4 is

$$-2\frac{N^{k+l+m+1}\big(|h|^2+|h'|^2/2\big)J_{klm}}{256\pi^6|x|^{2(k+l+m)}}\left(\frac{1}{\epsilon}+2+3\gamma_E+3\log\pi+3\log|x|^2\right) \tag{3.11}$$

We can see that the coefficients of $\log|x|^2$ (as well as that of $1/\epsilon$) in eqs. (3.5)–(3.7), (3.11) add up to zero. In particular, the term involving G_{klm} appears only from the contributions from figure 1 and 2 and they cancel. The term involving J_{klm} adds up to give a term proportional to $(g^2-|h|^2-|h'|^2/2)$, which is proportional to the beta function and hence vanishes in the conformal limit. Hence the only contribution to the required correlator comes from the F-term interaction as expected.

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3.3 F-term contribution

The computation of the anomalous dimension for quadratic operators such as $\mathrm{Tr}(Z_2 Z_3)$ and $\mathrm{Tr}(Z_1^2)$ differs from all other values of k, l, m. Postponing the details for quadratic operators, in the following subsection we will exclude values of k, l, m where $k+l+m=2$.

The contribution from the F-term is obtained from a diagram similar to the one given in figure 2. The interaction vertices involved are $|h|^2\mathrm{Tr}\big([Z_1, Z_2]_q[\bar{Z}_1, \bar{Z}_2]_q\big)$, $-|h'|^2\mathrm{Tr}\big(\bar{Z}_1{}^2 Z_1^2\big)$ and its cyclic permutations, where $[Z_1, Z_2]_q = qZ_1Z_2 - \bar{q}Z_2Z_1$. The combinatorics and integrals are the exactly as described earlier. In addition, here, when $k \neq 0, l = 1, m = 0$, contributions involving the parameter q appear. The factor S_{klm} is introduced to take this into account. The contribution to the above two-point correlator is

$$\Big[2|h|^2\ \Big((q^2+\bar{q}^2)S_{klm} + G_{klm}\Big) -\ 2|h'|^2(J_{klm}+G_{klm})\Big] \times \frac{N^{k+l+m+1}}{256\pi^6|x|^{2(k+l+m)}}\left(\frac{1}{\epsilon} + 1 + 5\gamma_E + 3\log(\pi) + 3\log|x|^2\right) \tag{3.12}$$

where

$$\begin{aligned} S_{klm} =\ & \delta_{k-1}\delta_l\big(1-\delta_m\big) + \delta_k\delta_{l-1}\big(1-\delta_m-\delta_{m-1}\big) \\ & +\delta_{l-1}\delta_m\big(1-\delta_k\big) + \delta_l\delta_{m-1}\big(1-\delta_k-\delta_{k-1}\big) \\ & +\delta_{m-1}\delta_k\big(1-\delta_l\big) + \delta_m\delta_{k-1}\big(1-\delta_l-\delta_{l-1}\big) \end{aligned} \tag{3.13}$$

The vanishing of the anomalous dimension now gives the condition

$$|h|^2\Big((q^2+\bar{q}^2)S_{klm} + G_{klm}\Big) -\ |h'|^2(J_{klm}+G_{klm})\Big) =\ 0 \tag{3.14}$$

Before looking for solutions in full generality, let us first consider the $\mathcal{N}=4$ limit, i.e., $q = \pm 1$ and $h' = 0$. In this limit, we obtain

$$2S_{klm} + G_{klm} =\ 0 \tag{3.15}$$

This has two non-trivial solutions:

(i) $k > 2$, $l = m = 0$ and permutations thereof;

(ii) $k > 1$, $l = 1$, $m = 0$ and permutations thereof.

(i) corresponds to operators of the form $\mathrm{Tr}\big(Z_1^k\big)$, and (ii) corresponds to operators of the form $\mathrm{Tr}\big(Z_1^k Z_2\big)$. All these are the known $\mathcal{N}=4$ chiral primary operators of the form we considered.[7]

We next consider the β-deformed theory which corresponds to keeping $h' = 0$ and restoring arbitrary values for q. The vanishing of the anomalous dimension is now

$$|h|^2\Big((q^2+\bar{q}^2)S_{klm} + G_{klm}\Big)\ =\ 0 \tag{3.16}$$

[7] This list actually misses out the quadratic operators $\mathrm{Tr}\big(Z_iZ_j\big)$ which are also chiral primaries. As mentioned earlier, the general formula given in eq. (3.14) is not valid for these operators since there is an extra contribution appearing in the deformed theory. This will be discussed in the next subsection.

Among the two classes of solutions that we obtained in the $\mathcal{N}=4$ limit, we see that those of type (i) continue to have vanishing anomalous dimension since S_{klm} and G_{klm} vanish separately for those values of k,l,m. However, this is no longer true for the operators of type (ii). These operators have the charges given in the list of chiral primaries given by Lunin and Maldacena for the β-deformed theory [11].

Finally, we now consider the Leigh-Strassler theory, where $h' \neq 0$ as well. None of the $\mathcal{N}=4$ chiral primaries are protected in this theory. However, in the limit $h=0$, the operator $\mathrm{Tr}\big(Z_1Z_2Z_3\big)$ (and also $\mathrm{Tr}\big(Z_1Z_3Z_2\big)$) is found to be a solution of the eq. (3.14), which is easy to understand as this operator cannot get any contribution at one loop from the h' interaction. We will now discuss the operators $\mathrm{Tr}\big(Z_1^2\big)$ and $\mathrm{Tr}\big(Z_2Z_3\big)$ that were not considered earlier. We will see that both these operators are protected in the Leigh-Strassler theory.

3.4 Anomalous dimension for $\mathbf{Tr}\big(Z_iZ_j\big)$

The anomalous dimension for dimension two operators obtains contributions from interactions involving double trace operators that appear in V_F. In the β-deformed theory, this interaction only affects the $\mathrm{Tr}(Z_2Z_3)$ operator, while in the general LS theory, the operator $\mathrm{Tr}\big(Z_1^2\big)$ is affected by the h' dependent double trace operator. For all other operators, one finds that interactions involving double trace operators provide contributions that are suppressed by a factor of $1/N$ relative to the single trace interactions and thus can be ignored in the large N limit.

The computation for these operators differs from the above due to a subtlety in taking the large N limit. The deformed theories have an extra interaction, as seen from eq. (2.2), which is suppressed by a factor of N relative to other interactions as it is a multi-trace operator.[8] For a dimension two operator, the trace algebra works out as follows,

$$\frac{1}{N}\,\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^cT^d)\mathrm{Tr}(T^cT^d) = \frac{1}{N}\left[\frac{N^2-1}{N}\times N\right]^2 = \frac{(N^2-1)^2}{N} \sim N^3\,. \tag{3.17}$$

The $\sim N^3$ contribution is seen to be of the same order as the one from the single trace interaction piece in eq. (2.2). This can be ignored in the large N limit while computing anomalous dimension for operators of dimension > 2. The important point to note is that this contribution is precisely the one that makes the anomalous dimension for $\mathrm{Tr}(Z_iZ_j)$ vanish (as has already been shown by others using different methods.)

3.5 Summary of results

We have seen that in the β-deformed theory, at one-loop, the family of operators of the form $\mathrm{Tr}\big(Z_1^k\big)$ and $\mathrm{Tr}\big(Z_1Z_2\big)$ are protected. For the Leigh-Strassler theory, we have only two kinds of operators which survive on including the h' deformation. They are $\mathrm{Tr}\big(Z_1^2\big)$ and $\mathrm{Tr}\big(Z_2Z_3\big)$.

---

[8] Recall that while the superfield Lagrangian has a single trace, the component Lagrangian is obtained by eliminating the auxiliary variables D and F. Thus, the bosonic potential ends up being a double trace which can be sometimes rewritten as a single trace using identities such as eq. (2.1). This term does *not* appear for $\mathrm{U}(N)$ as well. Note also that it vanishes in the $\mathcal{N}=4$ limit.

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In order to further generalise the kinds of operators that one must consider, we revisit the chiral primaries of the $\mathcal{N}=4$ theory. We have found chiral primaries that involve only two flavors of the scalars. What about those involving three flavours? The simplest one will involve one of each flavor. The $\mathcal{N}=4$ chiral primary is

$$\mathrm{Tr}\Big(Z_1Z_2Z_3+Z_1Z_3Z_2\Big)\ .$$

More generally, chiral primaries of $\mathcal{N}=4$ SYM are obtained by considering linear combinations of all possible orderings of operators. For instance, the above operator has $2=3!/3$ possibilities. The 3! is the order of the permutation group in 3 objects and the division by 3 reflects the cyclic property of the trace. Given a monomial, $z_1^{J_1}z_2^{J_2}z_3^{J_3}$, the corresponding $\mathcal{N}=4$ primary is given by the expression (with $n=J_1+J_2+J_3$)

$$\sum_{\pi\in S_n} c_\pi\ \mathrm{Tr}\Big(\pi Z_1^{J_1}Z_2^{J_2}Z_3^{J_3}\Big)\ , \tag{3.18}$$

where we sum over all permutations π and c_π is a symmetry factor [17]. Thus, there is a one-to-one correspondence between monomials and $\mathcal{N}=4$ chiral primaries. Thus, at dimension Δ_0, the number of chiral primaries is $(\Delta_0{+}1)(\Delta_0{+}2)/3$ which is the number of monomials at degree Δ_0. Based on the form of F-term equations like $\bar{F}_1=qZ_2Z_3-\bar{q}Z_3Z_2=0$, Freedman and Gürsoy(FG) have argued that one needs to associate a factor of $\bar{q}^2$ for terms that are related by the exchange of Z_2 and Z_3. For instance, the chiral primary involving all three flavors, will become

$$\mathrm{Tr}\Big(qZ_1Z_2Z_3+\bar{q}Z_1Z_3Z_2\Big)\ .$$

in the β-deformed theory by their prescription. We will refer to this as the FG prescription. In the sequel, we will verify the FG prescription works for operators involving up to six powers of the scalars only when they turn out to be chiral primaries.

4. Chiral primaries in the β-deformed theory

Chiral primaries in the β-deformed theory are classified by three charges corresponding to a $\mathrm{U}(1)^3$ subgroup of the SO(6) R-symmetry in the $\mathcal{N}=4$ theory. The three scalars Z_1, Z_2 and Z_3 have charges $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. Chiral primaries are thus labelled by their $\mathrm{U}(1)^3$ charges. Here we consider two-point functions of operators with charges (J_1,J_2,J_3): $(2,1,1)$, $(3,1,1)$, $(2,2,1)$, $(4,1,1)$ and $(2,2,2)$ — these are all the operators with $(J_1+J_2+J_3)\leq 6$ with all J_i non-vanishing.

For a given choice of (J_1,J_2,J_3), there are several operators that carry this charge. For example, there are three operators with charge $(2,1,1)$ as shown below. We choose linear combinations of all such operators in two steps. First, we obtain the corresponding $\mathcal{N}=4$ primary. Second, we introduce powers of $\bar{q}$ following the FG prescription [17]. This is potentially a candidate for a chiral primary. We then put in arbitrary coefficients in front of all operators to make our ansatz more general. We then compute the anomalous dimensions of these operators at planar one-loop and obtain the condition for the vanishing of their anomalous dimensions.

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In the following, we will see that the planar one-loop contribution to the anomalous dimension for all operators can be written as the sum of the absolute squares. This enables us to solve the equations easily without any hidden assumptions.

$(2,1,1)$ **operator**

We take the chiral primary to be of the following form

$$\mathcal{O}_{211} = \mathrm{Tr}\Big(Z_1^2 Z_2 Z_3 + b\bar{q}^2 Z_1 Z_2 Z_1 Z_3 + c\bar{q}^2 Z_2 Z_1^2 Z_3\Big) \ . \tag{4.1}$$

The computation of the anomalous dimension of this composite operator proceeds as before. The vanishing of the anomalous dimension is given by the condition[9]

$$\Big\{3|c|^2 + 4|b|^2 + 3 - 2\,\mathrm{Re}\Big[(2\bar{b} + \bar{c}) + 2\bar{q}^2 b\bar{c}\Big]\Big\} = 0 \tag{4.2}$$

The condition can be rewritten as follows:

$$2|b-1|^2 + |c-1|^2 + 2|b\bar{q}^2 - c|^2 \ = 0 \ .$$

This is a sum of three positive definite terms and is solved by $b = c = 1$ and $\bar{q}^2 = 1$ which makes it a chiral primary only for the $\mathcal{N} = 4$ theory.

$(3,1,1)$ **operator**

Here the chiral operator is taken to be

$$\mathcal{O}_{311} = \mathrm{Tr}\Big(Z_1^3 Z_2 Z_3 + b\bar{q}^2 Z_1^3 Z_3 Z_2 + c\bar{q}^2 Z_1^2 Z_2 Z_1 Z_3 + d\bar{q}^4 Z_1^2 Z_3 Z_1 Z_2\Big) \ . \tag{4.3}$$

There is an ambiguity in applying the FG prescription. For instance, the operator with coefficient b can be associated with either $\bar{q}^2$ (as we have chosen) or $\bar{q}^6$. This ambiguity disappears when $q^4 = 1$. The condition for the vanishing of the planar one-loop anomalous dimension is

$$3|b|^2 + 4|c|^2 + 4|d|^2 + 3 - 2\,\mathrm{Re}\Big[\bar{b} + 2\bar{c} + 2b\bar{d}q^4 + 2c\bar{d}\Big] = 0 \ . \tag{4.4}$$

The above expression can be written as follows:

$$|b-1|^2 + 2|c-1|^2 + 2|bq^4 - d|^2 + 2|c-d|^2 = 0 \ . \tag{4.5}$$

This has a solution only when $q^4 = 1$ and $b = c = d = 1$. This is precisely the situation where the FG prescription works. When $q = \pm 1$, this is $\mathcal{N} = 4$ chiral primary. When $q = \pm i$, this operator is also a protected operator. This is again a result expected from Lunin and Maldacena [11] — this is a chiral primary in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ theory. For all other values of β, this operator is not a chiral primary.

[9] It is easy to extract the 3×3 matrix of anomalous dimensions from the following expression. It may have some use in writing out the Hamiltonian for the spin-chain but its use is limited due to the small length of the chain.

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$(2,2,1)$ operator

For the operator

$$\mathcal{O}_{221} = \mathrm{Tr}\Big(Z_1^2 Z_2^2 Z_3 + b\bar{q}^2 Z_1^2 Z_2 Z_3 Z_2 + c\bar{q}^4 Z_1^2 Z_3 Z_2^2 + d\bar{q}^2 Z_1 Z_2 Z_3 Z_1 Z_2 \\ + f\bar{q}^4 Z_2 Z_1 Z_3 Z_2 Z_1 + g\bar{q}^4 Z_1 Z_2^2 Z_1 Z_3\Big)\,. \tag{4.6}$$

the condition for the vanishing of the one-loop anomalous dimension is

$$-2\mathrm{Re}\Big[\bar{b} + \bar{d} + \bar{g}q^2 + b\bar{c} + b\bar{f} + 2d\bar{f} + d\bar{g} + \big(b\bar{d} + c\bar{f} + f\bar{g} + c\bar{g}\big)q^2\Big] \\ +4|b|^2 + 3|c|^2 + 5|d|^2 + 5|f|^2 + 4|g|^2 + 3 = 0\,. \tag{4.7}$$

This can be written as the sum of squares as follows:

$$|b-1|^2 + |d-1|^2 + |gq^2-1|^2 + |b-c|^2 + |bq^2-d|^2 + |b-f|^2 \\ +|cq^2-g|^2 + |cq^2-f|^2 + 2|d-f|^2 + |d-g|^2 + |fq^2-g|^2 = 0\,. \tag{4.8}$$

The solution occurs only when $q^2 = 1$ and $b = c = d = f = g = 1$ which is the known $\mathcal{N} = 4$ chiral primary. Thus, this is not a chiral primary for generic values of β.

$(4,1,1)$ operator

We next consider the operator with charge $(4,1,1)$. Below the powers of q have been assigned using the FG prescription. However, there is an ambiguity in assigning the powers of $\bar{q}$. For instance, the operator multiplying the coefficient b_3 can be assigned either 1 or $\bar{q}^6$ since it can be reached by two different set of exchanges. This ambiguity however goes away when $q^6 = 1$.

$$\mathcal{O}_{411} = \mathrm{Tr}\Big(bZ_1^4 Z_2 Z_3 + b_1\bar{q}^2 Z_1^4 Z_3 Z_2 + b_2\bar{q}^4 Z_1^2 Z_2 Z_1^2 Z_3 \\ + b_3\bar{q}^6 Z_1 Z_2 Z_1^3 Z_3 + b_4\bar{q}^2 Z_1 Z_3 Z_1^3 Z_2\Big)\,. \tag{4.9}$$

The vanishing of the anomalous dimension at one-loop is

$$3|b|^2 + 3|b_1|^2 + 4|b_2|^2 + 4|b_3|^2 + 4|b_4|^2 - 4\mathrm{Re}\left[b\bar{b}_4 + \frac{1}{2}b\bar{b}_1 + b_1\bar{b}_3 q^6 + b_2\bar{b}_3 + b_2\bar{b}_4\right] = 0\,.$$

Again, this can be written as the sum of absolute squares.

$$|b-b_1|^2 + 2|b-b_4|^2 + 2|b_1q^6 - b_3|^2 + 2|b_2-b_3|^2 + 2|b_2-b_4|^2 = 0\,. \tag{4.10}$$

Clearly this has a solution $b = b_1 = b_2 = b_3 = b_4$ only when $q^6 = 1$. Note that this is precisely the value of q, where the ambiguity in the FG prescription is removed.

$(2,2,2)$ operator

For the operator

$$\begin{aligned}\mathcal{O}_{222}=\mathrm{Tr}\Big(&dZ_1^2Z_2^2Z_3^2+d_1\bar{q}^2Z_1^2Z_2Z_3Z_2Z_3+d_2\bar{q}^4Z_1^2Z_2Z_3^2Z_2+d_3\bar{q}^4Z_1^2Z_3Z_2^2Z_3\\&+d_4\bar{q}^6Z_1^2Z_3Z_2Z_3Z_2+d_5\bar{q}^8Z_1^2Z_3^2Z_2^2+d_6\bar{q}^2Z_3^2Z_1Z_2Z_1Z_2+d_7\bar{q}^4Z_1Z_2Z_1Z_3Z_2Z_3\\&+d_8\bar{q}^6Z_3^2Z_2Z_1Z_2Z_1+d_9\bar{q}^4Z_1Z_2^2Z_1Z_3^2+d_{14}\bar{q}^6Z_2^2Z_1Z_3Z_1Z_3\\&+\frac{d_{11}}{2}\bar{q}^2Z_1Z_2Z_3Z_1Z_2Z_3+d_{12}\bar{q}^4Z_2Z_1Z_2Z_3Z_1Z_3+d_{13}\bar{q}^4Z_1Z_3Z_1Z_2Z_3Z_2\\&+d_{10}\bar{q}^2Z_2^2Z_3Z_1Z_3Z_1+\frac{d_{15}}{2}\bar{q}^6Z_1Z_3Z_2Z_1Z_3Z_2\Big)\end{aligned} \tag{4.11}$$

we obtain the following condition for the vanishing of the anomalous dimension

$$\begin{aligned}&3|d|^2+5|d_1|^2+4|d_2|^2+4|d_3|^2+5|d_4|^2+3|d_5|^2+5|d_6|^2+6|d_7|^2+5|d_8|^2+4|d_9|^2\\&+3|d_{11}|^2+5|d_{14}|^2+6|d_{12}|^2+6|d_{13}|^2+5|d_{10}|^2+3|d_{15}|^2-2\mathrm{Re}\Big[d\bar{d}_6+d\bar{d}_1+d\bar{d}_{10}\\&+d_1\bar{d}_7+d_2\bar{d}_6+d_2\bar{d}_8+d_4\bar{d}_7+d_5\bar{d}_8+d_{14}\bar{d}_{12}+d_6\bar{d}_9+d_8\bar{d}_9+d_1\bar{d}_3+d_{13}\bar{d}_{10}+d_{11}\bar{d}_{13}\\&+d_{12}\bar{d}_{10}+d_{15}\bar{d}_{13}+d_3\bar{d}_4+d_{14}\bar{d}_{13}+d_5\bar{d}_4+d_6\bar{d}_7+d_1\bar{d}_2+d_4\bar{d}_2+d_{11}\bar{d}_{12}+d_7\bar{d}_8\\&+d_{15}\bar{d}_{12}+d_6\bar{d}_{12}+d_9\bar{d}_{10}+d_9\bar{d}_{14}+d_8\bar{d}_{12}+d_4\bar{d}_{13}+d_5\bar{d}_{14}+d_{10}\bar{d}_3+d_3\bar{d}_{14}\\&+d_{11}\bar{d}_7+d_{13}\bar{d}_1+d_{15}\bar{d}_7\Big]=0\,.\end{aligned} \tag{4.12}$$

This is independent of $q=e^{i\pi\beta}$ and is solved by $d=d_1=\cdots=1$ as can be clearly seen after rewriting the above expression in terms of sums of absolute squares.

$$\begin{aligned}&|d-d_1|^2+|d-d_6|^2+|d-d_{10}|^2+|d_1-d_7|^2+|d_1-d_3|^2+|d_1-d_2|^2+|d_1-d_{13}|^2\\&+|d_2-d_6|^2+|d_2-d_8|^2+|d_2-d_4|^2+|d_3-d_4|^2+|d_3-d_{14}|^2+|d_3-d_{10}|^2\\&+|d_5-d_8|^2+|d_5-d_4|^2+|d_5-d_{14}|^2+|d_4-d_7|^2+|d_4-d_{13}|^2+|d_6-d_9|^2\\&+|d_6-d_7|^2+|d_6-d_{12}|^2+|d_7-d_8|^2+|d_7-d_{11}|^2+|d_7-d_{15}|^2+|d_8-d_9|^2\\&+|d_8-d_{12}|^2+|d_9-d_{14}|^2+|d_9-d_{10}|^2+|d_{11}-d_{13}|^2+|d_{11}-d_{12}|^2+|d_{14}-d_{12}|^2\\&+|d_{14}-d_{13}|^2+|d_{12}-d_{10}|^2+|d_{12}-d_{15}|^2+|d_{13}-d_{10}|^2+|d_{13}-d_{15}|^2=0\,.\end{aligned} \tag{4.13}$$

Hence this is a chiral primary for any value of β on implementing the FG prescription.

By studying the non-renormalisation properties of operators up to dimension six, we see that for generic β, chiral primaries appear only as operators of the form (k,k,k) and $(k,0,0)$ (other than the quadratic operators) as expected from the refs. [17, 14, 15, 11]. Further, the FG prescription works for these operators. The absence of an ambiguity in implementing the FG prescription seems to be the key to the vanishing of the one-loop anomalous dimension. This also picks out the special values of q for which some operators are protected.

5. General Leigh-Strassler deformation

The general Leigh-Strassler deformation is invariant under the action of the trihedral group $\Delta(27)$ which is a finite non-abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$. The centre of this group is a $\mathbb{Z}_3$ which is sub-group of $\mathrm{U}(1)_R$. Thus, the $\mathrm{U}(1)_R$ charge can be identified with the $\mathbb{Z}_3$ charge. Chiral primaries in this theory must appear as irreducible representations of $\Delta(27)$. In appendix B, we have provided relevant details of the irreducible representations of $\Delta(27)$. Based on the representation theory, we obtain the following important and useful result:

1. When the scaling dimension, $\Delta_0 = 0$ mod 3, then chiral primaries *must* appear in any one of the nine one-dimensional representations, $\mathcal{L}_{Q,j}$ $(Q, j = 0, 1, 2)$. The representation $\mathcal{L}_{0,0}$ corresponds to a singlet of $\Delta(27)$. We will label such operators $\mathcal{O}^{(Q,j)}_{\Delta_0}$ to indicate the representation they belong to. The charge Q for one-dimensional representations can be identified with the charge proposed in [21].

2. When the scaling dimension, $\Delta_0 = a$ mod 3 $(a \neq 0)$, then chiral primaries appear in the three-dimensional representation, $\mathcal{V}_a$ and thus three operators form a triplet. We label all such operators by $\mathcal{O}^a_{\Delta_0}$. Given one operator of the triplet, the other two can be generated by the cyclic replacement $\tau : Z_1 \to Z_2 \to Z_3 \to Z_1$.

This observation is useful in many ways. There will be no mixing between operators which sit in distinct representations of $\Delta(27)$. This leads to a nine-fold reduction in the operators that one needs to consider for one-dimensional representations and a three-fold reduction for the three-dimensional representations.

$\Delta_0 = 3$, $Q = 0$ operators

Since $\Delta_0 = 0$ mod 3, one has to only consider the one-dimensional representations. There are three operators with $(Q, j) = (0, 0)$ and we will consider the most general linear combination of them.

$$\mathcal{O}^{(0,0)}_3 = \mathrm{tr}\Big(aZ_1^3 + aZ_2^3 + aZ_3^3 + bZ_1Z_2Z_3 + cZ_1Z_3Z_2\Big) \tag{5.1}$$

The vanishing of the one-loop correction to the anomalous dimension is given by

$$\begin{aligned}27|a|^2|h'|^2 + 9(h\bar{h}'qa\bar{b} + \bar{h}h'\bar{q}\bar{a}b) - 9(h\bar{h}'\bar{q}a\bar{c} + \bar{h}h'q\bar{a}c)\\ -3(|h|^2\bar{q}^2b\bar{c} + |h|^2q^2\bar{b}c) + 3(|b|^2 + |c|^2)|h|^2 = 0\end{aligned} \tag{5.2}$$

This can easily be seen as equivalent to

$$|\bar{h}\Big(b\bar{q} - cq\Big) + 3a\bar{h}'|^2 \;=\; 0 \tag{5.3}$$

This has two solutions:

(i) $a = 0$, $b = q$ and $c = \bar{q}$. This implies that the $\mathrm{Tr}\big(qZ_1Z_2Z_3 + \bar{q}Z_1Z_3Z_2\big)$ which was a chiral primary in the β-deformed theory is protected at one-loop in the LS theory as well.

(ii) $a = 1$, $b = -\frac{3\bar{h}'}{2\bar{h}}q$, $c = \frac{3\bar{h}'}{2\bar{h}}\bar{q}$. This is the operator

$$\mathrm{Tr}\Big[(Z_1^3 + Z_2^3 + Z_3^3) - \frac{3m}{2}(qZ_1Z_2Z_3 - \bar{q}Z_1Z_3Z_2)\Big]\,, \text{ where } \; m \equiv \frac{\bar{h}'}{\bar{h}}\;.$$

There are two other operators with $Q = 0$ and $j = 1, 2$ — these are

$$\mathcal{O}_3^{(0,j)} = \mathrm{Tr}\big[Z_1^3 + \omega^j Z_2^3 + \omega^{2j} Z_3^3\big]\;.$$

These are descendants[10] and hence are not chiral primaries.

$\Delta_0 = 3$, $Q = 1$ operators

For this operator

$$\mathcal{O}_3^{(1,j)} = \mathrm{Tr}\Big(Z_1^2 Z_2 + \omega^j Z_2^2 Z_3 + \omega^{2j} Z_3^2 Z_1\Big) \tag{5.4}$$

the vanishing of the one-loop correction to the anomalous dimension is

$$3\Big(|h'|^2 + |h|^2|q-\bar{q}|^2\Big) \;+\; 2\mathrm{Re}\Big[h\bar{h}'(q-\bar{q})\Big(1+\omega^j+\omega^{2j}\Big)\Big] = 0 \tag{5.5}$$

When $j \neq 0$, the above equation has no solution (except in the $\mathcal{N} = 4$ limit) implying that the operators are not chiral primaries. They are known descendants [21]. However, when $j = 0$, the condition becomes

$$3|h' + h(q-\bar{q})|^2 = 0\,,$$

which has a solution only when $h' = h(q - \bar{q})$. At all other points in the space of couplings, the operator is a descendant.

$\Delta_0 = 4$ operator

The operator that we will consider here is in the three-dimensional representation, $\mathcal{V}_1$, of $\Delta(27)$. Below, we will consider only one operator in the triplet since the result is valid for all three operators. The operator has charge $Q = 1$. Further, it is a linear combination of several $\mathcal{N} = 4$ primaries — the unknown coefficients are labelled to remind the reader of this fact. For instance, below terms with coefficient using the same letter of the alphabet are part of the same $\mathcal{N} = 4$ primary.

$$\begin{aligned}\mathcal{O}_4^1 = \mathrm{tr}\Big(&Z_2^4 + b\; Z_1^3 Z_2 + c\; Z_1^2 Z_3^2 + c_1 Z_1 Z_3 Z_1 Z_3 + d\; Z_1 Z_2^2 Z_3 \\ &+ d_1 Z_1 Z_2 Z_3 Z_2 + d_2 Z_3 Z_2^2 Z_1 + f Z_3^3 Z_2\Big)\end{aligned} \tag{5.6}$$

[10] It follows from the representation theory of $\Delta(27)$ that there are nine descendants (with $\Delta_0 = 3$), one in each of the irreps, $\mathcal{L}_{Q,j}$. When, $(Q, j) = (0, 0)$, there are three operators and one descendant while there are one operator in other sectors. We obtain two protected operators in the $\Delta_0 = 3$ sector which is consistent with this counting.

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Requiring the vanishing of the one-loop correction to the anomalous dimension, we get,

$$
\begin{aligned}
16|h'|^2 &+ |b|^2\Big(2|h'|^2+|h|^2|q-\bar{q}|^2\Big) + 2|c|^2\Big(|h'|^2+|h|^2\Big) + |d|^2\Big(|h'|^2+3|h|^2\Big) \\
&+ |f|^2\Big(2|h'|^2+|h|^2|q-\bar{q}|^2\Big) + 4|h|^2|d_1|^2 + 8|h|^2|c_1|^2 + |d_2|^2\Big(|h'|^2+3|h|^2\Big) \\
+ 2\mathrm{Re}\Big[&4h\bar{h}'q\,\bar{d} - 4h\bar{h}'\bar{q}\,\bar{d}_2 - h'\bar{h}(q-\bar{q})b\bar{c} + h\bar{h}'q\,b\bar{d} - h'\bar{h}(q-\bar{q})d_1\bar{b} - h\bar{h}'\bar{q}\,b\bar{d}_2 \\
&- h'\bar{h}q\,c\bar{d} - 2|h|^2(c\bar{c}_1)(q^2+\bar{q}^2) + h\bar{h}'q\,d_2\bar{c} + h\bar{h}'(q-\bar{q})c\bar{f} + 2h'\bar{h}\bar{q}\,c_1\bar{d} \\
&- 2h'\bar{h}q\,c_1\bar{d}_2 + h'\bar{h}\bar{q}\,d\bar{f} - h'\bar{h}(q-\bar{q})d_1\bar{f} - h'\bar{h}q\,d_2\bar{f} - 2|h|^2q^2\,d_1\bar{d} \\
&- |h|^2q^2\,d_2\bar{d} - 2|h|^2\bar{q}^2\,d_1\bar{d}_2\Big] = 0\,.
\end{aligned} \tag{5.7}
$$

The above equation is rather hard to analyse and one may wonder if it has a solution. We now make use of the fact that in the limit $h' = 0$, these equations should provide us conditions that appeared in the β-deformed theory. We have already seen that these can be written as the sum of squares. Using this result as input (and a check!), we deform the $h' = 0$ term suitably such that all terms that appear as $h'\bar{h}$ that appear above are accounted for. This strategy works rather well and we obtain an expression (given below) that is easily analysed.

$$
\begin{aligned}
&|4\bar{h}' + \bar{h}\bar{q}d - \bar{h}qd_2|^2 + |\bar{h}\bar{q}d - \bar{h}qd_1 + \bar{h}'b|^2 + |\bar{h}\bar{q}d - \bar{h}qd_1 + \bar{h}'f|^2 \\
&+ |\bar{h}\bar{q}d_1 - \bar{h}qd_2 + \bar{h}'b|^2 + |\bar{h}\bar{q}d_1 - \bar{h}qd_2 + \bar{h}'f|^2 + |\bar{h}\bar{q}c - 2\bar{h}qc_1 - \bar{h}'d|^2 \\
&+ |\bar{h}qc - 2\bar{h}\bar{q}c_1 + \bar{h}'d_2|^2 + |\bar{h}(q-\bar{q})b - \bar{h}'c|^2 + |\bar{h}(q-\bar{q})f - \bar{h}'c|^2 = 0
\end{aligned} \tag{5.8}
$$

This equation has the following definite solution providing us the required chiral primary operator at one-loop planar level.

$$
\begin{aligned}
b = f &= \frac{4m^3\,(q^2+\bar{q}^2-1)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} & c &= \frac{4m^2\,(q-\bar{q})\,(q^2+\bar{q}^2-1)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} \\
d &= -\frac{4m\,\big(m^3\bar{q}-q(q-\bar{q})^2\big)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} & c_1 &= \frac{2m^2\,\big(q-\bar{q}+m^3\big)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} \\
d_1 &= \frac{4m\,(q-\bar{q})\big(q-\bar{q}+m^3\big)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} & d_2 &= \frac{4m\,\big(q^2+\bar{q}^2-2+m^3q^3\big)}{q^2(m^3(q^2+\bar{q}^2)-(q-\bar{q})^3)}
\end{aligned} \tag{5.9}
$$

where $m = \frac{\bar{h}'}{\bar{h}}$. We thus find only one protected operator that exists for generic values of the couplings.

$\Delta_0 = 6$, $Q = 1$ operators

We consider dimension six operators which are in the one-dimensional representation of $\Delta(27)$. This is an example where the condition that the operator be in an irrep of $\Delta(27)$ (rather than an abelian subgroup as considered in [18], for instance) leads to a simplification. There is a three-fold reduction in the number of constants in the problem.

$$\begin{aligned}\mathcal{O}_6^{(1,j)} = \mathrm{Tr}\Big(& Z_1^5 Z_2 + b Z_1^4 Z_3^2 + b_1 Z_1^3 Z_3 Z_1 Z_3 + b_2 Z_1^2 Z_3 Z_1^2 Z_3 + c Z_1^3 Z_2^2 Z_3 \\ & + c_1 Z_1^3 Z_2 Z_3 Z_2 + c_2 Z_1^3 Z_3 Z_2^2 + c_3 Z_1^2 Z_2 Z_1 Z_2 Z_3 + c_4 Z_1^2 Z_2 Z_1 Z_3 Z_2 \\ & + c_5 Z_1^2 Z_2^2 Z_1 Z_3 + c_6 Z_1^2 Z_2 Z_3 Z_1 Z_2 + c_7 Z_1^2 Z_3 Z_1 Z_2^2 + c_8 Z_1^2 Z_3 Z_2 Z_1 Z_2 \\ & + c_9 Z_1 Z_2 Z_1 Z_2 Z_1 Z_3 \Big) + \omega^j \mathrm{Tr}\Big(Z_2^5 Z_3 + b Z_2^4 Z_1^2 + \cdots \Big) + \omega^{2j} \mathrm{Tr}\Big(Z_3^5 Z_1 + b Z_3^4 Z_2^2 + \cdots \Big)\end{aligned} \tag{5.10}$$

Given the complexity of the expression for the anomalous dimension, we used the same strategy that was employed for the $\Delta_0 = 4$ operator. This enabled us to re-express the anomalous dimension as the sum of absolute squares as given below.[11]

$$\begin{aligned}& |(q-\bar{q}) - mb|^2 + |m + \bar{q}c_3 - qc_4|^2 + |m + \bar{q}c - qc_1|^2 + |m + \bar{q}c_1 - qc_2|^2 \\ & + |m + \bar{q}c_6 - qc_8|^2 + |mb + \bar{q}c - qc_5|^2 + |mc_7 + \bar{q}c - qc_3|^2 + |mc + \bar{q}c_6 - qc_1|^2 \\ & + |mc_2 + \bar{q}c_1 - qc_4|^2 + |mc_5 + \bar{q}c_8 - qc_2|^2 + |mb + \bar{q}c_7 - qc_2|^2 + |mb_1 + \bar{q}c_3 - qc_9|^2 \\ & + |mc + \bar{q}c_3 - qc_5|^2 + |mc_6 + \bar{q}c_3 - qc_6|^2 + |mc_1 + \bar{q}c_9 - qc_4|^2 + |mb + \bar{q}c_6 - qc_4|^2 \\ & + |mc_4 + \bar{q}c_4 - qc_8|^2 + |2mb_2 + \bar{q}c_5 - qc_7|^2 + |mc_8 + \bar{q}c_5 - qc_9|^2 + |mc_1 + \bar{q}c_6 - qc_9|^2 \\ & + |mc_3 + \bar{q}c_9 - qc_7|^2 + |mc_2 + \bar{q}c_7 - qc_8|^2 + |mb_1 + \bar{q}c_9 - qc_8|^2 + |mc + \bar{q}b_1 - qb|^2 \\ & + |mc_2 + \bar{q}b - qb_1|^2 + |mc_5 + 2\bar{q}b_2 - qb_1|^2 + |mc_7 + \bar{q}b_1 - 2qb_2|^2 = 0\end{aligned} \tag{5.11}$$

Trying to solve the constraint equations arising from the above, we can see that there is no generic state satisfying them. Thus, there are no protected operators for generic values of couplings. However, at specific sub-loci in the coupling space there are solutions. These belong to several branches which we list below: Branches (i) and (ii) are connected to the operators that appear when $q^4 = 1$ in the β-deformed theory. Branch (v) degenerates to a linear combination of $\mathcal{N} = 4$ primaries when $q^2 = 1$. Branches (iii) and (iv) do not have such a limit.

(i) $q^2 = -1$, with h, h' arbitrary. The solution is given by $b = (q - \bar{q})/m$, $b_1 = (m^2 + 2)/mq$, $b_2 = q/m$; $c = c_2 = c_4 = c_5 = c_6 = c_7 = 1$, $c_1 = c_8 = -(1+mq)$, $c_3 = q(q+m)$ and $c_9 = (mq - 1 - m^2)$.

(ii) $q^2 = 1$, $mq = -1$. The solution is given by the choices $b = c_1 = c_8 = 0$, $b_1 = b_2 = c = c_4 = c_5 = c_6 = c_9 = 1$, $c_2 = c_7 = -1$ and $c_3 = 2$.

(iii) $m = 1/q$. The solution is given by $b = c_1 = c_3 = q^2 - 1$, $b_1 = c_2 = c_4 = c_6 = c_7 = c_9 = 1$, $b_2 = 1/q^2$, $c = q^4 - q^2 + 1$, $c_5 = (q^4 - 2)/q^2$ and $c_8 = 2/q^2$.

(iv) $m = -q$. The solution is given by $b = q^{-2} - 1$, $b_1 = c = c_4 = c_5 = ic_6 = c_9 = 1$, $b_2 = q^2$, $c_1 = -1$, $c_2 = q^{-4} - q^{-2} - 1$, $c_3 = 2q^2$, $c_7 = q^{-2} - 2q^2$ and $c_8 = -1 + q^{-2}$.

(v) $m = q - \bar{q}$, $b = b_1 = c = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = 1$ and $b_2 = 1/2$.

[11] The corresponding expression for the $j \neq 0$ operators take a similar form. It is obtained by the the following replacements on the $j = 0$ expressions: $mb \to mb\omega^j$, $mb_i \to mb_i\omega^j$, $mc \to mc\omega^{2j}$. and $mc_i \to mc_i\omega^{2j}$. Note that if any of the coefficients appears without being multiplied by m, it is left unchanged. It turns out these equations do not have a solution indicating that they are all descendants.

$\Delta_0 = 6$, $Q = 0$ operators

We now consider operators with $\Delta_0 = 6$ and $Q = 0$. There are three operators in the representations, $\mathcal{L}_{0,j}$ that we need to consider. We first consider the operator in the representation $\mathcal{L}_{0,0}$ — it consists of 46 terms — however, the number of independent coefficients is reduced to 26 due to the use of the trihedral symmetry. We write the operator as the sum of four terms

$$\mathcal{O}_6^{(0,0)} = \mathcal{O}_1 + \tau\big(\mathcal{O}_1\big) + \tau^2\big(\mathcal{O}_1\big) + \mathcal{O}_2\,, \tag{5.12}$$

where

$$\begin{aligned}
\mathcal{O}_1 =& \mathrm{Tr}\Big(aZ_1^6 + bZ_1^4Z_2Z_3 + b_1Z_1^4Z_3Z_2 + b_2Z_1^2Z_2Z_1^2Z_3 + b_3Z_1Z_2Z_1^3Z_3 \\
&+ b_4Z_1^3Z_2Z_1Z_3 + cZ_1^3Z_2^3 + c_1Z_1^2Z_2Z_1Z_2^2 + c_2Z_1^2Z_2^2Z_1Z_2 + c_3Z_1Z_2Z_1Z_2Z_1Z_2\Big)\,, \\
\mathcal{O}_2 =& \mathrm{Tr}\Big(d\,Z_1^2Z_2^2Z_3^2 + d_1Z_1^2Z_2Z_3Z_2Z_3 + d_2Z_1^2Z_2Z_3^2Z_2 + d_3Z_1^2Z_3Z_2^2Z_3 \\
&+ d_4Z_1^2Z_3Z_2Z_3Z_2 + d_5Z_1^2Z_3^2Z_2^2 + d_6Z_3^2Z_1Z_2Z_1Z_2 + d_7Z_1Z_2Z_1Z_3Z_2Z_3 \\
&+ d_8Z_3^2Z_2Z_1Z_2Z_1 + d_9Z_2^2Z_1Z_3^2Z_1 + d_{10}Z_2^2Z_3Z_1Z_3Z_1 + d_{11}Z_1Z_2Z_3Z_1Z_2Z_3 \\
&+ d_{12}Z_1Z_2Z_3Z_1Z_3Z_2 + d_{13}Z_1Z_2Z_3Z_2Z_1Z_3 + d_{14}Z_2^2Z_1Z_3Z_1Z_3 + d_{15}Z_1Z_3Z_2Z_1Z_3Z_2\Big)\,,
\end{aligned}$$

and by $\tau(\mathcal{O}_1)$ we mean the operator obtained by the cyclic replacement $\tau : Z_1 \to Z_2 \to Z_3 \to Z_1$ in all the terms. The operators in the representations $\mathcal{L}_{0,j}$ $(j \neq 0)$ are given by

$$\mathcal{O}_6^{(0,j)} = \mathcal{O}_1 + \omega^j\tau(\mathcal{O}_1) + \omega^{2j}\tau^2(\mathcal{O}_1)\ . \tag{5.13}$$

After some long and rather tedious algebra, one can express the vanishing of the one-loop anomalous dimension for the operator $\mathcal{O}_6^{(0,0)}$ as the sum of absolute squares.

$$\begin{aligned}
&3\,|6am - b_1q + b\bar{q}|^2 + |b_3m - d_7q + d_6\bar{q}|^2 + |b_4m - d_8q + d_7\bar{q}|^2 + |b_4m - d_4q + d_{13}\bar{q}|^2 \\
&+|b_4m - d_4q + d_7\bar{q}|^2 + |b_3m - d_7q + d_1\bar{q}|^2 + |b_1m - 2d_{15}q + d_7\bar{q}|^2 + |bm - d_7q + 2d_{11}\bar{q}|^2 \\
&+|b_3m - d_{13}q + d_{10}\bar{q}|^2 + |b_4m - d_{14}q + d_{13}\bar{q}|^2 + |bm - d_{13}q + 2d_{11}\bar{q}|^2 \\
&+|b_1m - 2d_{15}q + d_{13}\bar{q}|^2 + |b_3m - d_{13}q + d_1\bar{q}|^2 + |b_1m - 2d_{15}q + d_{12}\bar{q}|^2 \\
&+|bm - d_{12}q + 2d_{11}\bar{q}|^2 + |b_3m - d_{12}q + d_{10}\bar{q}|^2 + |b_4m - d_{14}q + d_{12}\bar{q}|^2 \\
&+|b_4m - d_8q + d_{12}\bar{q}|^2 + |b_3m - d_{12}q + d_6\bar{q}|^2 + |bm - d_9q + d_{10}\bar{q}|^2 \\
&+|b_1m - d_{14}q + d_9\bar{q}|^2 + |b_1m - d_8q + d_9\bar{q}|^2 + |b_2m - d_5q + d_8\bar{q}|^2 \\
&+|b_1m - d_8q + d_2\bar{q}|^2 + |b_1m - d_{14}q + d_3\bar{q}|^2 + |bm - d_3q + d_{10}\bar{q}|^2 \\
&+|b_2m - d_{10}q + d\bar{q}|^2 + |b_2m - d_1q + d\bar{q}|^2 + |bm - d_2q + d_1\bar{q}|^2 \\
&+|b_2m - d_6q + d\bar{q}|^2 + |bm - d_3q + d_1\bar{q}|^2 + |bm - d_2q + d_6\bar{q}|^2 + |b_1m - d_4q + d_2\bar{q}|^2 \\
&+|b_1m - d_4q + d_3\bar{q}|^2 + |b_2m - d_5q + d_4\bar{q}|^2 + |bm - d_9q + d_6\bar{q}|^2 \\
&+|d_6m - c_1q + 3c_3\bar{q}|^2 + |d_8m - 3c_3q + c_2\bar{q}|^2 + |d_{14}m - 3c_3q + c_2\bar{q}|^2 \\
&+3|dm - cq + c_2\bar{q}|^2 + 3|d_5m - c_1q + c\bar{q}|^2 + 2|d_2m - c_2q + c_1\bar{q}|^2 \\
&+2|d_9m - c_2q + c_1\bar{q}|^2 + |d_{10}m - c_1q + 3c_3\bar{q}|^2 + 2|d_3m - c_2q + c_1\bar{q}|^2 \\
&+|d_4m - 3c_3q + c_2\bar{q}|^2 + |d_1m - c_1q + 3c_3\bar{q}|^2 + 6|cm - b_4q + b\bar{q}|^2 \\
&+6|c_1m - b_2q + b_4\bar{q}|^2 + 6|cm - b_1q + b_3\bar{q}|^2 + 6|c_2m - b_3q + b_2\bar{q}|^2 = 0
\end{aligned} \tag{5.14}$$

The above equations lead to 52 equations in 26 unknowns. We find that there are precisely *two* solutions that exists for generic values of the parameters. It is easy to see that there are some identifications amongst the d_i. They are $d_1 = d_6 = d_{10}$, $d_2 = d_3 = d_9$, $d_4 = d_8 = d_{14}$ and $d_7 = d_{12} = d_{13}$. This reduces the number of unknowns to 18. The explicit solutions are somewhat unilluminating and will not be presented here. An important point is that the two solutions can be characterised by $a = 0$ and $a \neq 0$. In the limit, $h' \to 0$, these two solutions reduce to the operators that exist in the β-deformed theory for general values of β. This is precisely what happened for the $\Delta_0 = 3$, $Q = 0$ operators as well. This is a clear indication that the operators that are protected at one-loop in the β-deformed theory and are in the representation $\mathcal{L}_{0,0}$ survive the h' deformation. The operators in the representation $\mathcal{L}_{0,j}$ with $j \neq 0$ are however not protected operators.

5.1 Summary of results

We have shown that it is useful to organise chiral primaries in terms of representations of the discrete group $\Delta(27)$. In particular, we have seen that operators appearing only in one of the three representations, $\mathcal{L}_{0,0}$ and $\mathcal{V}_a$ are protected at planar one-loop level. We conjecture that this result is true in general. The general pattern for operators protected at planar one-loop (and possibly beyond) organised in terms of the trihedral group is given in the table below when $\Delta_0 > 2$. (We have excluded the quadratic operators since they have a somewhat different behaviour.)

Scaling dim.	$\mathcal{N} = 4$ theory	β-def. theory	LS theory
$\Delta_0 = 3r$	$\mathcal{L}_{0,0} \oplus \frac{r(r+1)}{2} \left[\oplus_{i,j} \mathcal{L}_{i,j} \right]$	$\mathcal{L}_{0,0} \oplus_j \mathcal{L}_{0,j}$	$2\mathcal{L}_{0,0}$
$\Delta_0 = a \bmod 3$	$\frac{(\Delta_0+1)(\Delta_0+2)}{6} \mathcal{V}_a$	$\mathcal{V}_a$	$\mathcal{V}_a$

The first column is only a reorganisation of the well-understood $\mathcal{N} = 4$ primaries into representations of $\Delta(27)$ [17]. The second column follows from the Lunin-Maldacena prediction that chiral primaries in the β-deformed theory, for generic values of β, arise only with charges (k, k, k) and $(k, 0, 0)$ rewritten in terms of representations of $\Delta(27)$ [11]. The last column is based on our computations in the LS theory and has been verified up to and including scaling dimension six.

Further, we have seen that in other representations, operators are only protected in a submanifold in the space of couplings. These submanifolds consist of several branches, some of which do not intersect the subspace of β-deformed theories.

6. Conclusion

In this paper, we have studied the planar one-loop contribution of operators in the LS theory for operators up to dimension six. We have used the trihedral group to classify the operators and this has lead to a significant simplification to the problem. We find that for generic values of couplings, the protected operators arise in the one-dimensional representation, $\mathcal{L}_{0,0}$ when $\Delta_0 = 0 \bmod 3$ and in the three-dimensional representations $\mathcal{V}_a$

when $\Delta_0 = a \bmod 3$ ($a = 1, 2$). We conjecture that this is true in general. It is interesting to see if there is a simple proof of this statement.

The Leigh-Strassler superpotential makes an interesting appearance in a different context. A recent computation all-orders perturbative computation of the effective superpotential for the so-called long-branes on the cubic torus using the topological Landau-Ginzburg model turns out to be of the Leigh-Strassler form [32]. It is possible that this computation may be related to the quantum effective superpotential of the LS theory.[12] We are pursuing the relationship of this work to the LS theory [34]. In particular, even if a direct map doesn't exist, it suggests a re-ordering of the perturbative computation of the quantum effective superpotential for the LS theory and that the renormalised coefficients (up to an overall normalisation) should be expressible in terms of theta functions of characteristic three. This statement is modulo the effect of the the chiral Konishi anomaly which may modify the statement.

An open question is to find the gravity duals for LS theories. A more limited question is to ask whether one can find special values of the couplings like the case of rational β in the β-deformed theory. The crucial input in finding the gravity duals whenever β was rational is the realisation that the effect of discrete torsion in abelian orbifolds is to q-deform the $\mathcal{N}=4$ superpotential [12 – 15]. One may ask whether discrete torsion in non-abelian orbifolds could also produce the h' deformation. The naive answer based on adapting the analysis of ref. [6] to include discrete torsion is that no such couplings can arise. However, since those results are based on 'dimensional reduction', it would be interesting to actually carry out a CFT computation in string theory to verify that such terms are not generated to come up with a no-go theorem.

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A. LS deformed $\mathcal{N}=4$ Yang-Mills theory

The Lagrangian density of the Leigh-Strassler theory in terms of $\mathcal{N}=1$ superfields is

$$\begin{aligned}\mathcal{L} = &\int d^2\theta d^2\bar{\theta}\,\mathrm{Tr}\Big(e^{-gV}\bar{\Phi}e^{gV}\Phi\Big) + \bigg[\frac{1}{2g^2}\int d^2\theta\,\mathrm{Tr}\Big(W^\alpha W_\alpha\Big) \qquad \text{(A.1)}\\ &+ih\int d^2\theta\,\mathrm{Tr}\Big(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\Big) + \frac{ih'}{3}\int d^2\theta\,\mathrm{Tr}\Big(\Phi_1^3+\Phi_2^3+\Phi_3^3\Big) + c.c.\bigg]\end{aligned}$$

[12] The authors of ref. [33] obtain an expression for the quantum effective superpotential for the β-deformed theory using a relationship with matrix models. It is also of interest to see if these two effective potentials are related.

We denote the lowest component of the superfield Φ_i by Z_i and its fermionic partner by ψ_i. The vector multiplet in the Wess-Zumino gauge has as components, the gauge field, A_μ and its superpartner, the gaugino, λ in addition to the auxiliary field D. All fields transform in the adjoint of SU(N). Writing these in component fields we get the Lagrangian

$$\begin{aligned}\mathcal{L} =& \mathrm{Tr}\Big(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\; 2\bar{\lambda}\sigma^\mu D_\mu\lambda - i\;\bar{\psi}_i\sigma^\mu D_\mu\psi_i + D^\mu Z_i D_\mu \bar{Z}_i \\ & - \frac{g^2}{4}[Z_i,\bar{Z}_i][Z_j,\bar{Z}_j] + i\sqrt{2}g\;\bar{\psi}_i[Z_i,\bar{\lambda}] + i\sqrt{2}g\;\psi_i[\bar{Z}_i,\lambda] - ih\;\bar{\psi}_3[Z_1,\psi_2] \\ & + i\bar{h}\;\bar{\psi}_3[\bar{Z}_1,\psi_2] - ih' Z_1\psi_1\psi_1 + i\bar{h}'\bar{Z}_1\bar{\psi}_1\bar{\psi}_1 + \text{cyc. perm.}\Big) - V_F(Z) \end{aligned} \tag{A.2}$$

where $D_\mu Z_i = \partial_\mu Z_i + ig[A_\mu, Z_i]$ and $V_F(Z)$ is as given in eq. (2.2).

A.1 Trace formulae for SU(N)

Below, we provide the trace identities and normalisations that we have used in our paper.

$$\begin{aligned} T^aT^a = \frac{N^2-1}{N}I \qquad\qquad & \mathrm{Tr}(T^aT^b) = \delta^{ab} \\ \mathrm{Tr}(AT^aBT^a) = \mathrm{Tr}(A)\mathrm{Tr}(B) - & \frac{1}{N}\mathrm{Tr}(AB) \\ \mathrm{Tr}(AT^a)\mathrm{Tr}(BT^a) = \mathrm{Tr}(AB) - & \frac{1}{N}\mathrm{Tr}(A)\mathrm{Tr}(B) \end{aligned} \tag{A.3}$$

B. Representations of the trihedral group, $\Delta(27)$

In this appendix, we discuss the representation theory of the trihedral group $\Delta(27)$. [36–38] We expect chiral primaries of the Leigh-Strassler deformed theory to be in irreducible representations of this group. Trihedral groups are finite subgroups of $SL(3,\mathbb{C})$ of the form $A \rtimes \mathcal{C}_3$, where A is a diagonal abelian group and $\mathcal{C}_3$ is the cyclic $\mathbb{Z}_3$ generated by

$$\tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

For the trihedral group, $\Delta(27)$, the group A is generated by $g = \frac{1}{3}(1,1,1)$ and $h = \frac{1}{3}(0,1,-1)$.[13] In our example, g turns out to be the centre of $\Delta(27)$ and is a subgroup of $U(1)_R$.

The irreducible representations of $\Delta(27)$ consist of nine one-dimensional representations, $\mathcal{L}_{Q,j}$ $(Q, j = 0,1,2)$ and two three-dimensional representations $\mathcal{V}_a$ $(a = 1,2)$. The charge under g can be clearly identified with $U(1)_R$ charge.

$\mathcal{L}_{\mathbf{Q,j}}$ In the one-dimensional representations, one has the following action of the generators h and τ

$$h\cdot v = \omega^Q\; v\,, \quad \tau\cdot v = \omega^j\; v \qquad \text{where } v\in\mathcal{L}_{Q,j} \text{ and } \omega = e^{2\pi i/3}\ .$$

[13] We denote by $\frac{1}{R}(a,b,c)$ the matrix $\mathrm{Diag}(\epsilon^a,\epsilon^b,\epsilon^c)$ with ϵ, a non-trivial R-th root of unity.

$\mathcal{V}_\mathbf{a}$ In the three-dimensional representation, one has

$$h \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^a & 0 \\ 0 & 0 & \omega^{2a} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \quad \text{and } \tau \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

for $a = 1, 2$. Note that when $a = 0$, the above representation is reducible to a direct sum, $\oplus_{j=0}^{2} \mathcal{L}_{1,j}$, of one-dimensional representations.

The chiral superfields (Φ_1, Φ_2, Φ_3) are in the representation $\mathcal{V}_1$ while their anti-chiral partners transform in the representation $\mathcal{V}_2$. We will choose our chiral primaries to be in one of these representations. Since the interactions in the Leigh-Strassler theory are invariant under $\Delta(27)$, it follows that chiral operators in distinct representations of $\Delta(27)$ cannot mix. However, operators in the same representation can and do mix.

B.1 Polynomials as irreps of $\Delta(27)$

On taking the commutative limit, all our chiral primary operators reduce to polynomials in three variables, (z_1, z_2, z_3), where we replaced the matrices Z_i by scalars z_i (using the lower-case to indicate the replacement). First, the triplet $(z_1, z_2, z_3)^T$ transforms in the representation, $\mathcal{V}_1$. Second, we can organise the polynomials by degree — the degree is the (naive) scaling dimension of the corresponding operator which we denote by Δ_0. Thirdly, Δ_0 mod 3, is the $\mathbb{Z}_3$ charge of the polynomial under the centre of the group (which is generated by g defined above).

Polynomials in these variables of a given degree, Δ_0, can be further organised into irreducible representations of $\Delta(27)$. The precise representation is decided by the value of Δ_0 mod 3. One has the following result:

- [$\mathbf{\Delta_0 = 0}$ **mod 3**] All polynomials can be organised in one-dimensional representations of Δ(27, i.e., $\mathcal{L}_{Q,j}$. For example, when $\Delta_0 = 3$ the polynomials $(z_1^3 + \omega^j z_2^3 + \omega^{2j} z_3^3) \in \mathcal{L}_{0,j}$ and $z_1 z_2 z_3 \in \mathcal{L}_{0,0}$. In particular, there is no polynomial whose degree is 0 mod 3 that is in the representation $\mathcal{V}_1$ or $\mathcal{V}_2$.

- [$\mathbf{\Delta_0 \neq 0}$ **mod 3**] All polynomials must necessarily arise in one of the the three-dimensional representations. In fact, defining $a = \Delta_0$ mod 3 with $a = 1$ or 2, the polynomials must be in the three-dimensional representation $\mathcal{V}_a$. For example, consider the F-term equations ($dW = 0$) which are of degree two. A straightforward analysis shows that the three equations are in the representation $\mathcal{V}_2$.

The proof of the above statements goes as follows. Note that the generator g can be realised in terms of h and τ as $h\tau^{-1}h^2\tau$. Using this, notice that for a vector $v \in \mathcal{L}_{Q,j}$, $g \cdot v = \omega^{3Q} v = v$. Thus, the $\mathbb{Z}_3$ charge associated with g is zero implying that the $\mathrm{U}(1)_R$ charge is zero modulo three. Similarly, for a triplet $\vec{v} \in \mathcal{V}_a$, $g \cdot \vec{v} = \omega^a \, \vec{v}$ implying that the $\mathrm{U}(1)_R$ charge is a modulo three.

This leads to the following conclusion: All chiral primaries with $\Delta_0 = 0$ mod 3 must be in any one of the one-dimensional representations of $\Delta(27)$ while those where $\Delta_0 = a$ mod 3 must arise in the three dimensional representation $\mathcal{V}_a$.

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C. Integrals

We evaluate dimensionally regulated momentum integrals, with $D = 4 - 2\epsilon$, using the Feynman parametrisation

$$\begin{aligned}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2)^{\alpha_1}(p^2)^{\alpha_2}} &= \frac{1}{B[\alpha_1,\alpha_2]}\int_0^1 dx \int_0^1 dy\ \delta(1-x-y)\ x^{\alpha_1-1}y^{\alpha_2-1} \\ &\qquad\qquad \times\int \frac{d^Dk}{(2\pi)^D}\frac{1}{(xk^2+yp^2)^{\alpha_1+\alpha_2}} \\ &= \frac{1}{B[\alpha_1,\alpha_2]}\int_0^1 dx x^{\alpha_1-1}(1-x)^{\alpha_2-1}\times\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(xk^2+(1-x)p^2)^{\alpha_1+\alpha_2}}\end{aligned} \tag{C.1}$$

where $B[\alpha_1,\alpha_2]$ is the beta-function.

The Fourier transform to position space is done using the following integral.

$$\int \frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot x}}{(p^2)^s} = \frac{\Gamma[\frac{D}{2}-s]}{4^\epsilon\ \pi^{\frac{D}{2}}\Gamma[s](x^2)^{\frac{D}{2}-s}} \tag{C.2}$$

Momentum integral (i).

$$\begin{aligned}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k-p)^2} &= \int_0^1 dx\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\big((k-px)^2+p^2x(1-x)\big)^2} \\ &= \int_0^1 dx\frac{\Gamma(\epsilon)x^{-\epsilon}(1-x)^{-\epsilon}}{(4\pi)^{2-\epsilon}p^{2\epsilon}} = \frac{\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}p^{2\epsilon}}\end{aligned} \tag{C.3}$$

Using the formula (C.2) above we obtain the integral in eq. (3.2) in position space as

$$\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\frac{1}{(p^2)^{2\epsilon}} = \frac{\Gamma[2-3\epsilon]}{4^{2\epsilon}\ \pi^{2-\epsilon}\ \Gamma[2\epsilon]\ (|x|^2)^{2-3\epsilon}} \tag{C.4}$$

The integral in eq. (3.2) gives

$$\left(\frac{\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}p^{2\epsilon}}\right)^2\frac{\Gamma[2-3\epsilon]}{4^{2\epsilon}\ \pi^{2-\epsilon}\ \Gamma[2\epsilon]\ (|x|^2)^{2-3\epsilon}} \tag{C.5}$$

We expand this in powers of ϵ and obtain the answer in eq. (3.5).

Momentum integral (ii). The integral in eq. (3.11) using Feynman parametrisation

$$\begin{aligned}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(p-q)^2(p^2)^{1+\epsilon}} &= \frac{1}{B[1,1+\epsilon]}\int_0^1 dx\ x^\epsilon\int\frac{d^Dp}{(2\pi)^D}\frac{1}{\Big(x(p-q)^2+xp^2\Big)^{2+\epsilon}} \\ &= \int_0^1 dx\ x^\epsilon\Big(q^2x(1-x)\Big)^{-2\epsilon}\frac{\Gamma[2\epsilon]}{(4\pi)^{2-\epsilon}\ B[1,1+\epsilon]\Gamma[2+\epsilon]} \\ &= \frac{\Gamma[2\epsilon]B[1-2\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}\ B[1,1+\epsilon]\Gamma[2+\epsilon](q^2)^{-2\epsilon}}\end{aligned} \tag{C.6}$$

Again taking the Fourier transform of the above expression we get the expression in eq. (3.11).

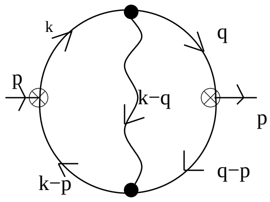

Figure 5: Gluon exchange

Gluon exchange contribution to anomalous dimensions. The contribution from interaction terms $ig\,\mathrm{Tr}(\partial_\mu Z_i[A^\mu, \bar{Z}_i])$ and $ig\,\mathrm{Tr}(\partial_\mu \bar{Z}_i[A^\mu, Z_i])$ to the anomalous dimension of the operator $\mathcal{O}$ is computed here. There are different contractions giving two distinct momentum integrals. The diagram in figure 5 can be evaluated when *Case(1):* Both the interaction vertices involve $\partial_\mu Z$ (or $\partial_\mu \bar{Z}$), *Case(2):* When one vertex involves $\partial_\mu Z$ and the other $\partial_\mu \bar{Z}$. We work out both the cases below.

Case(1) : From figure 5 given here we write down the Feynman integral for this case.

$$\int\int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{k\cdot(k-p)}{k^2(k-q)^2(k-p)^2 q^2 (q-p)^2} \tag{C.7}$$

Considering only the part which is divergent we get,

$$\int\int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{1}{(k-q)^2(k-p)^2 q^2 (q-p)^2}$$

We consider the k-integration first. Take $k' = k - p$. After dimensionally regulating and using Feynman parametrisation

$$\begin{aligned}
\int \frac{d^D k'}{(2\pi)^D} \frac{1}{k'^2(k'-q+p)^2} &= \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D} \frac{1}{\Big((k'-(q-p)x)^2 + (q-p)x(1-x)\Big)^2} \\
&= \frac{\Gamma[\epsilon]}{(4\pi)^{2-\epsilon}\Gamma[2]((q-p)^2)^\epsilon} \int_0^1 dx x^{-\epsilon}(1-x)^{-\epsilon} \\
&= \frac{\Gamma[\epsilon] B[1-\epsilon, 1-\epsilon]}{(4\pi)^{2-\epsilon}(q-p)^{2\epsilon}}
\end{aligned} \tag{C.8}$$

Doing the q-integration in the same way,

$$\begin{aligned}
\frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2\big((q-p)^2\big)^{1+\epsilon}} &= \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}} \frac{1}{B[1,1+\epsilon]} \\
&\quad \times \int_0^1 dy\, y^\epsilon \int \frac{d^D q}{(2\pi)^D} \frac{1}{\big((q-yp)^2 + p^2 y(1-y)\big)^{2+\epsilon}} \\
&= \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}} \frac{\Gamma[2\epsilon]}{(4\pi)^{2-\epsilon}B[1,1+\epsilon]\Gamma[2+\epsilon](p^2)^{2\epsilon}} \\
&\quad \times \int_0^1 dy\; y^{-\epsilon}(1-y)^{-2\epsilon} \\
&= \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{4-2\epsilon}} \frac{\Gamma[2\epsilon]B[1-\epsilon,1-2\epsilon]}{\Gamma[1+\epsilon](p^2)^{2\epsilon}}
\end{aligned} \tag{C.9}$$

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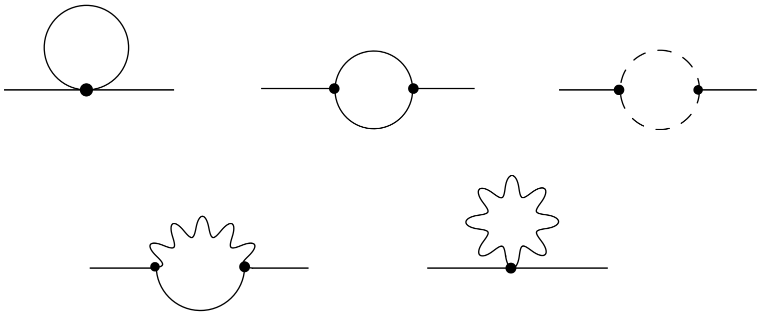

Figure 6: One loop corrections to the scalar propagator

Again using formula (C.2) we Fourier transform and expand in powers of ϵ to obtain

$$\frac{1}{256\pi^6|x|^4}\left(\frac{1}{\epsilon}+2+3\gamma_E+3\log(\pi)+3\log(|x|^2)\right) \tag{C.10}$$

Case(2) :

$$\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k\cdot(q-p)}{k^2(k-q)^2(k-p)^2q^2(q-p)^2} \tag{C.11}$$

The divergent part of this is

$$\begin{aligned}
&\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k\cdot q}{k^2(k-q)^2(k-p)^2q^2(q-p)^2}\\
&=\frac{1}{2}\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k^2+q^2-(k-q)^2}{k^2(k-q)^2(k-p)^2q^2(q-p)^2}\\
&=\frac{1}{2}\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\left[\frac{1}{(k-q)^2(k-p)^2q^2(q-p)^2}+\frac{1}{k^2(k-q)^2(k-p)^2(q-p)^2}\right.\\
&\qquad\left.-\frac{1}{k^2(k-q)^2q^2(q-p)^2}\right]\\
&=\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\left[\frac{1}{(k-q)^2(k-p)^2q^2(q-p)^2}-\frac{1}{2k^2(k-q)^2q^2(q-p)^2}\right]
\end{aligned} \tag{C.12}$$

The two integrals in the final expression above are already evaluated.

$$\left(\frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]\Gamma[2\epsilon]B[1-\epsilon,1-2\epsilon]}{\Gamma[1+\epsilon]}-\frac{\Gamma^4[1-\epsilon]\Gamma^2[\epsilon]}{2\;\Gamma^2[2-2\epsilon]}\right)\frac{1}{(4\pi)^{4-2\epsilon}(p^2)^{2\epsilon}} \tag{C.13}$$

Fourier transforming to position space using formula (C.2) and expanding in powers of ϵ, we get the value of the above integral to be $\frac{1}{256\pi^6|x|^2}$. Hence there is no contribution from this integral to the anomalous dimension.

D. One loop correction to scalar propagator

The diagrams in figure 6 are the one-loop contributions to the two point function $\langle Z_1\bar{Z}_1\rangle$. The dashed lines are the fermionic propagators and the wiggly lines are gauge boson propagators. Gauge boson interactions are calculated in Landau gauge.

Contribution from scalar tadpole due to interaction term $-\frac{g^2}{4}\mathrm{tr}([Z_i,\bar{Z}_i][Z_j,\bar{Z}_j])$

$$-\frac{g^2}{4}\cdot 2\cdot \mathrm{tr}\Big((T^bT^c-T^cT^b)(T^cT^a-T^aT^c)\Big)\int d^Dk\frac{1}{p^2k^2p^2} = -g^2\cdot\Big(Ntr(T^aT^b)\Big)\int d^Dk\frac{1}{p^2k^2p^2}$$
$$= -g^2Ntr(T^aT^b)\int d^Dk\frac{1}{p^2k^2p^2} \tag{D.1}$$

where p_μ is the external momentum and $D = 4-2\epsilon$.

Contribution from scalar tadpole due to interaction terms in $V_F(Z)$

$$4\Big[|h|^2\,\mathrm{tr}\Big((qT^bT^c-\bar{q}T^cT^b)(qT^aT^c-\bar{q}T^cT^a)\Big)+|h'|^2\mathrm{tr}\big(T^bT^cT^d\big)\big(T^dT^cT^b\big)\Big]\times\int d^Dk\frac{1}{p^2k^2p^2}$$
$$= -4(|h|^2+|h'|^2/2)Ntr(T^aT^b)\int d^Dk\;\frac{1}{p^2k^2p^2} \tag{D.2}$$

Contribution from Yukawa interaction vertices $i\sqrt{2}g\;\mathrm{tr}\big(\psi_i[\bar{Z}_i,\lambda]+\bar{\psi}_i[Z_i,\bar{\lambda}]\big)$.

The contribution from this interaction

$$-(i\sqrt{2})^2g^2\mathrm{tr}\big(T^cT^aT^d-T^cT^dT^a\big)\mathrm{tr}\big(T^cT^bT^d-T^cT^dT^b\big)\int d^Dk\frac{\sigma^\mu k_\mu\sigma^\nu(p-k)_\nu}{2p^2k^2(p-k)^2p^2}$$
$$= -4g^2\;Ntr(T^aT^b)\int d^Dk\frac{k\cdot p-k^2}{p^2k^2(p-k)^2p^2} \tag{D.3}$$

Here we also use the identity $\mathrm{tr}(\sigma^\mu k_\mu\sigma^\nu p_\nu) = 2k\cdot p$

Contribution from Yukawa interaction vertices $-ih\;\mathrm{tr}\big(\psi_3[Z_1,\psi_2]_q\big)-ih'Z_1\psi_1\psi_1+$c.c.)

$$\Big[-(i)^2|h|^2\mathrm{tr}\big(qT^cT^bT^d-\bar{q}T^cT^dT^b\big)\mathrm{Tr}\big(qT^cT^aT^d-\bar{q}T^cT^dT^a\big)+|h'|^2\mathrm{tr}\big(T^bT^cT^d\big)\big(T^dT^cT^b\big)\Big]$$
$$\times\int d^Dk\frac{\sigma^\mu k_\mu\sigma^\nu(p-k)_\nu}{p^2k^2(p-k)^2p^2} \tag{D.4}$$
$$= -4(|h|^2+|h'|^2/2)\;Ntr(T^aT^b)\int d^Dk\frac{k\cdot p-k^2}{p^2k^2(p-k)^2p^2}$$

Contribution from scalar-gluon interaction terms ig $\mathrm{tr}(\partial_\mu Z_i[A_\mu,\bar{Z}_i])$, ig $\mathrm{tr}(\partial_\mu\bar{Z}_i[A_\mu,Z_i])$

$$2(i)^2g^2\;\mathrm{tr}\big(T^bT^cT^d-T^bT^dT^c\big)\;\mathrm{tr}\big(T^aT^cT^d-T^aT^dT^c\big)\int d^Dk\Big[\frac{p^2}{2p^2k^2(p-k)^2p^2}\Big]$$
$$+2(i)^2g^2\;\mathrm{tr}\big(T^dT^cT^a-T^dT^aT^c\big)\;\mathrm{tr}\big(T^dT^cT^b-T^dT^bT^c\big)\int d^Dk\Big[\frac{(p-k)^2}{2p^2k^2(p-k)^2p^2}\Big]$$
$$+4\;(i)^2g^2\mathrm{tr}\big(T^dT^cT^a-T^dT^aT^c\big)\cdot\mathrm{tr}\big(T^bT^cT^d-T^bT^dT^c\big)\int d^Dk\Big[\frac{p^2-p\cdot k}{2p^2k^2(p-k)^2p^2}\Big]$$
$$=\;2g^2N\mathrm{tr}(T^aT^b)\int d^Dk\Big[\frac{1}{2p^2\;k^2\;p^2}+\frac{1}{p^2k^2(k-p)^2p^2}\Big] \tag{D.5}$$

Contribution from gluon tadpole due to interaction term $-g^2\mathrm{tr}([A_\mu,Z_i][A_\mu,\bar{Z}_i])$

$$-g^2\cdot\mathrm{tr}\Big((T^cT^b-T^bT^c)(T^cT^a-T^aT^c)\Big)\int d^Dk\frac{g_{\mu\nu}g^{\mu\nu}}{p^2k^2p^2} = -4g^2N\mathrm{tr}(T^aT^b)\int d^Dk\frac{1}{p^2k^2p^2} \tag{D.6}$$

Summing the contributions from each diagram, we see that quadratic divergences cancel. The one-loop correction to $\langle Z_i^a \bar{Z}_j^b \rangle$ is given as

$$-(2N)\ \mathrm{tr}(T^a T^b) \cdot \big(|h|^2 + |h'|^2/2\big) \int d^D k\ \frac{1}{k^2 (p-k)^2 p^2} \tag{D.7}$$

Fourier transforming we get

$$= -N \cdot \mathrm{tr}(T^a T^b) \cdot \big(|h|^2 + |h'|^2/2\big)\ \big(Y_{122} + Y_{112}\big) \tag{D.8}$$

where $Y_{ijk} = \int d^4 x \frac{1}{(x-x_i)^2 (x-x_j)^2 (x-x_k)^2}$. In the large N limit we have $|h|^2 + |h'|^2/2 = g^2$. Hence we should get back the one-loop correction in $\mathcal{N} = 4$ theory. From eq. (D.7) above, we get,

$$2N \cdot g^2\ \mathrm{tr}(T^a T^b) \cdot \big(Y_{122} + Y_{112}\big) \tag{D.9}$$

This expression is similar in structure with the one obtained in [35].

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